Applications of Complex Networks

Javier M. Buldú
http://www.complex.etsit.urjc.es/jmbuldu
Complex Systems Group
Universidad Rey Juan Carlos (Madrid, Spain)
OUTLINE OF THE COURSE

0.- Bibliography

1.- Introduction to Complex networks
   1.1.- What is a (complex) network?
   1.2.- Types of networks
   1.3.- Basic concepts about networks
   1.4.- Brief historical background

2.- Applications of Complex Networks
   2.1.- Social Sciences
   2.2.- Technological Networks
   2.3.- Biological Networks

3.- Future trends (and paranoias!)
0. - BIBLIOGRAPHY

Review Articles about Complex Networks:


Review articles about REAL complex Networks:

0. - BIBLIOGRAPHY

Popular Science Books:


Complex Networks Databases:

- Mark Newman, University of Michigan: http://www-personal.umich.edu/~mejn/netdata/
- Alberto L. Barabási, University of Notre Dame: http://www.nd.edu/~networks/resources.htm
- Alex Arenas, Universitat Rovira y Virgili: http://deim.urv.cat/~aarenas/data/welcome.htm
- Indiana University databases: http://iv.slis.indiana.edu/db/index.html
1.- INTRODUCTION TO COMPLEX NETWORKS

1.1.- What is a (complex) network?
1.1.- WHAT IS A (COMPLEX) NETWORK?

- **A Network** is a set of elements with connections between them

A network (graph) $G=(N,M)$ consists of a set of $N=\{n_1, n_2, \ldots, n_N\}$ nodes and a set of $L=\{l_1, l_2, \ldots, l_M\}$ links.

A graph is the mathematical abstraction of a network. Despite it is not rigorous, we will use both terms, graph and network, as synonyms.

From this viewpoint, each element is represented by a site (physics), node (computer science), actor (sociology) or vertex (graph theory) and the interaction between two elements corresponds to a bond (physics), link (computer science), tie (sociology) or edge (graph theory).
1.1.- WHAT IS A (COMPLEX) NETWORK?

- Nodes and links may arise from completely different contexts:

Schematic representation of a network of hosts and routers.

Simplified representation of the Arctic food web.
1.1. WHAT IS A (COMPLEX) NETWORK?

**Applications of Complex Networks**

- **Metabolic network of the E. Coli.**
  - From Guimerà et al., *Nature*, 433, 895, 2005

- **Madrid Power Grid.**
  - From http://www.ree.es
1.1.- WHAT IS A (COMPLEX) NETWORK?

Structure of romantic and sexual contact at Jefferson High School
From P.S. Bearman et al., AJS, 110, 44 (2004)

Neuron network
1.1.- WHAT IS A (COMPLEX) NETWORK?

A Complex Network is a network with non-trivial topological features, with patterns of connection between their elements that are neither purely regular nor purely random.
1.2.- Types of networks
1.2.- TYPES OF NETWORKS

- There exist different classifications of networks:
  - According to the direction of the links: directed or undirected.
  - According to the kind of interaction: weighted or unweighted.
  - According to the differences between nodes: bipartite or not.
  - According to the evolution of their topology: static or evolving.
  - According to the dynamics of the nodes: with/without dynamics.
  - ...
1.2. - TYPES OF NETWORKS

- Directed and undirected networks:

  The relationship between nodes may be symmetric (undirected networks) or asymmetric (directed networks).

  **Undirected network**
  ![Undirected network diagram](image)
  Examples: router network, power grid, collaboration networks, etc...

  **Directed network (digraph)**
  ![Directed network diagram](image)
  Examples: internet, food webs, e-mail/telephone networks, etc...

  The direction of the links is crucial in dynamical processes occurring in the network, such as information spreading, synchronization or network robustness.

1.2.- TYPES OF NETWORKS

- **Weighted and unweighted networks:**

  The capacity or intensity of the relationship between nodes may be heterogeneous (weighted networks).

  **Unweighted network**

  ![Example of an unweighted network]

  **Weighted network**

  ![Example of a weighted network]

  Examples: citation network, internet, etc...

  Examples: e-mail/telephone networks, food webs, power grid, collaboration network, etc...

Again, the weight of the links is crucial in dynamical processes occurring in the network, such as information spreading, synchronization or network robustness.
1.2. - TYPES OF NETWORKS

- **Bipartite networks:**

  Networks with two (or more) kind of nodes and links joining ONLY nodes of unlike type.

![Diagram of Bipartite Networks]

Examples: recommendation networks, user-item based networks, etc...

Despite being bipartite, it is possible to project the network.
1.2. TYPES OF NETWORKS

- **Static or evolving networks:**

  Networks do not appear suddenly. We have to know if the network that we are studying is static (its structure is stationary) or if it is still evolving.

  Two fundamental questions are addressed when working with evolving networks: what are the rules governing the evolution? What consequences have the rules on the final topology?
1.2.- TYPES OF NETWORKS

- Networks of dynamical systems:

Nodes are dynamical systems whose dynamics is influenced through the matrix of connections.

Nodes are (coupled) dynamical systems (periodic oscillators, excitable systems, chaotic oscillators, bistable systems, ...)

\[
\dot{\phi}_i = \left\{ \begin{array}{ll}
\omega_i + \frac{d}{(k_i + k_p)} \sum_{j=1}^{N} a_{ij} \sin(\phi_j - \phi_i) \\
+ \frac{d_p k_{pi}}{(k_i + k_p)} \sin(\phi_{pi} - \phi_i),
\end{array} \right.
\]

In this case, we have to study the influence of the topology in the dynamical processes occurring in the network (synchronization, stochastic processes, etc..) ... ... and vice-versa!
Despite the different types of networks, which in turn are obtained from completely different interacting systems (people, neurons, proteins, routers, ...), we will see that they share some universal properties. Is it a social network? A technological network? A biological network?
1.3.- Basic concepts about networks
1.3.- BASIC CONCEPTS ABOUT NETWORKS

Adjacency, Weights and Laplacian Matrix:

All the former networks can be described using a matricial formalism. Given a set of $N$ nodes with $M$ connections between them:

Weights Matrix ($W$):
Entries of the matrix are the weights $w_{ij}$ ($i,j=1, \ldots, N$) of the connections

\[
\begin{pmatrix}
0.0 & 2.3 & 4.1 & 0.0 \\
2.3 & 0.0 & 1.0 & 0.0 \\
4.1 & 1.0 & 0.0 & 7.1 \\
0.0 & 0.0 & 7.1 & 0.0 \\
\end{pmatrix}
\]

Adjacency Matrix ($A$):
$a_{ij}=1$ if there exists a link between $i$ and $j$, and $a_{ij}=0$ otherwise

\[
\begin{pmatrix}
0 & 1 & 1 & 0 \\
1 & 0 & 1 & 0 \\
1 & 1 & 0 & 1 \\
0 & 0 & 1 & 0 \\
\end{pmatrix}
\]

Laplacian Matrix ($L$):
The Laplacian matrix is defined as $L=K-A$, where $K$ is a diagonal matrix of elements $k_{ii}=\sum a_{ij}$. Thus, it has a zero-row sum.

\[
\begin{pmatrix}
-2 & 1 & 1 & 0 \\
1 & -2 & 1 & 0 \\
1 & 1 & -3 & 1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}
\]

Matrices will be symmetric if networks are undirected.
1.3.- BASIC CONCEPTS ABOUT NETWORKS

- Shortest path, average path length and diameter:

  **Shortest path \((d_{ij})\):**
  
  The shortest path \(d_{ij}\) between nodes \(i\) and \(j\) corresponds to the minimal distance (or weight) between all paths that connect \(i\) and \(j\).

  **Average path length \((l)\):**
  
  The average path length \(l\) is the average shortest path between all nodes in the network:
  \[
  l = \langle d_{ij} \rangle = \frac{1}{N(N-1) \sum_{i \neq j} d_{ij}}
  \]
  
  when the network is not connected it is useful to define the “harmonic mean”
  \[
  l = \frac{1}{\langle d_{ij}^{-1} \rangle} = \left( \frac{1}{N(N-1) \sum_{i \neq j} \frac{1}{d_{ij}}} \right)^{-1}
  \]

  **Diameter\((D)\):**
  
  The maximum between all shortest paths \(D = \max(d_{ij})\)

  **Component:**
  
  The set of nodes reachable from a given node.
1.3.- BASIC CONCEPTS ABOUT NETWORKS

- Degree, strength and betweenness:

  **Degree \((k_i)\):**

  The degree \(k_i\) of a node \(i\) is the number of connections of the node.

  **Strength \((s_i)\):**

  The strength \(s_i\) of a node \(i\) is the sum of the weights of the connections to that node:

  \[ s_i = \sum w_{ij} \]

  **Betweenness \((b_i)\):**

  The betweenness of a node \(i\) (or a link) accounts for the number of shortest paths passing through that node (or link).

  \[
  b_i = \sum_{j,k \in \mathcal{V}, j \neq k} \frac{n_{jk}(i)}{n_{jk}}
  \]

  where \(n_{jk}\) is the number of shortest paths connecting \(j\) and \(k\), and \(n_{jk}(i)\) are those shortest paths between \(j\) and \(k\) that pass through \(i\).
1.3.- BASIC CONCEPTS ABOUT NETWORKS

Network Motifs:

Network motifs are patterns (sub-graphs) that recur within a network much more often than expected at random.

Example: all 13 types of three-node connected subgraphs:

Each network motif can carry out specific information-processing functions

1.3.- BASIC CONCEPTS ABOUT NETWORKS

- **Clustering coefficient:**

  The clustering coefficient $C$ accounts for the number of triangles in the network. Specifically, $C_i$ is the ratio between the number of links $E$ connecting the nearest neighbors of $i$ and the total number of possible links between these neighbors.

  $$C_i = \frac{2E}{k_i(k_i - 1)}$$

  The clustering coefficient of the network $C$ is the average of $C_i$ over all nodes.

  

  \[
  C_{1,2,3,4} = \{0,0,0,0\} \quad C=0
  \]

  \[
  C_{1,2,3,4} = \{1,1,1,1\} \quad C=1
  \]

  \[
  C_{1,2,3,4} = \{1,0,1,1/3\} \quad C=7/12
  \]
1.3. BASIC CONCEPTS ABOUT NETWORKS

- Local and Global Efficiency:

  The efficiency overcomes the divergence of the shortest paths if the graph is disconnected

  **Global Efficency \((E)\):**

  The global efficiency is the harmonic mean of the geodesic paths between all nodes of the network:

  \[
  E = \frac{1}{N(N-1)} \sum_{i,j \in N, i \neq j} \frac{1}{d_{ij}}
  \]

  **Local Efficency \((E_i)\):**

  The local efficiency \(E_i\) of a node \(i\), measures the shortest path length between the subset \(G_i\) of neighbors of the node \(i\), when \(i\) is not present.

  \[
  E_{loc} = \frac{1}{N} \sum_{i \in N} E(G_i)
  \]

  The local efficiency is related, somehow, with the clustering coefficient.
1.3.- BASIC CONCEPTS ABOUT NETWORKS

Graph Spectrum:

The spectrum of a graph is the set of eigenvalues of its adjacency (or Laplacian) matrix $A$. A graph $G_{N,M}$ has $N$ eigenvalues $\mu = (\mu_1, \mu_2, ..., \mu_N)$ and $N$ associated eigenvectors $v = (v_1, v_2, ..., v_N)$.

The eigenvalues and associated eigenvectors of a graph are intimately related to important topological features such as the diameter, the number of cycles, information transmission and the connectivity properties of the graph.

Spectral density:

$$\rho(\mu) = \frac{1}{N} \sum_{i=1}^{N} \delta(\mu - \mu_i)$$

Rescaled spectral density of three random graphs having $p=0.05$ and size $N=100$, $N=300$, and $N=1000$. The isolated peak corresponds to the principal eigenvalue.
### 1.3.- BASIC CONCEPTS ABOUT NETWORKS

#### Community Structure (I):

Given a graph $G_{N,M}$, a community is a subgraph $G'_{N',M'}$ whose nodes are tightly connected (or at least, more connected than in a random equivalent network).

---

Figure from: Guimerà et al., Nature, 433, 895(2005)

---

Zachary Karate Club

1.3. BASIC CONCEPTS ABOUT NETWORKS

- Community Structure (II):

Several algorithms have been proposed in order to split a sparse network into communities:

Modularity $M$ is an objective measure in order to evaluate community division:

$$ M = \sum_{s=1}^{N_M} \left[ \frac{l_s}{L} - \left( \frac{d_s}{2L} \right)^2 \right] $$

where $N_M$ is the number of modules, $L$ is the number of links in the network, $l_s$ is the number of links between nodes in module $s$, and $d_s$ is the sum of the degrees of the nodes in module $s$.

Figure from: L. Danon et al., World Scientific, 93-113 (2007)
1.3.- BASIC CONCEPTS ABOUT NETWORKS

- Degree Distribution (I):

The [cumulative] degree distribution \([P_c(k)]\) \(p(k)\) accounts for the fraction of nodes in the network with a degree [higher than] equal to \(k\).
1.3. BASIC CONCEPTS ABOUT NETWORKS

- Degree Distribution (II):

  Two types of degree distribution appear more frequently in real networks:

- Exponential decay: \( P_c(k) \sim e^{-\alpha k} \)

- Power-law decay: \( P_c(k) \sim k^{-\gamma} \)

  Typical in random networks

  Networks with power-law decay are called scale-free networks.
1.3.- BASIC CONCEPTS ABOUT NETWORKS

- **Degree Distribution (III):**

Other related distributions are:

- In/out degree distributions (directed networks)
- Strength distribution (weighted networks)

In/out degree distributions of WWW (from two different samples: 325,729 and 200,000,000 nodes). From R. Albert et al., Rev. Mod. Phys. 74, 47 (2002).

1.3.- BASIC CONCEPTS ABOUT NETWORKS

- **Clustering Distribution C(k):**

The clustering distribution has been related with the modularity and hierarchy of the network:

Figure: Clustering distribution in three organisms: *Aquidex aeolicus* (archaea) (C), *Escherichia coli* (bacterium) (D), and *Saccharomyces cerevisiae* (eukaryote) (E). (F) The C(k) curves averaged over all 43 organisms is shown, and the inset displays all 43 species together. Lines correspond to $C(k) \sim k^{-1}$, and diamonds represent the C(k) value expected for an equivalent scale-free network, indicating the absence of scaling.

1.3.- BASIC CONCEPTS ABOUT NETWORKS

- Nearest neighbor degree $k_{nn}(k)$ and assortativity

The $k_{nn}(k)$ distribution measures the degree of the nearest neighbors. It is an indicator of the assortativity of the network.

Collaboration and similarity network obtained from a music database (AllMusic Guide).
From J. Park et al., IJBC, 17, 2281 (2007).
1.4.- Brief historical background
1.4.- BRIEF HISTORICAL BACKGROUND

- **Leonard Euler (Basel 1707 - St. Petersburg 1783)**

Some revealing data about Leo:

- Euler worked in almost all areas of mathematics: geometry, calculus, trigonometry, algebra, and number theory, as well as continuum physics, lunar theory and other areas of physics.

- Large number of topics of physics and mathematics are named in his honour (e.g., Eulers’s function, Euler’s Equation or Euler’s formula).

- All his work is collected in *Opera Omnia*, which consists of 886 books.

- With one eye from 1738 and completely blind from 1766!

- And the most astonishing data: all of that with **13 children**!
1.4.- BRIEF HISTORICAL BACKGROUND

- Euler, the father of graph theory:

The seven bridges of Konigsberg and the origin of graph theory: Is it possible to cross the seven bridges only once?

Euler’s Solution:

\[ N_0 = \text{Number of nodes with odd degree} \]

1.- If \( N_0 > 2 \), no solution.

2.- If \( N_0 = 2 \), only one solution starting from one of the odd nodes.

1.- If \( N_0 < 2 \), there are solutions starting from any node.
1.4.- BRIEF HISTORICAL BACKGROUND

- Regular Graphs
  - After the death of Euler, graph theory received many contributions from mathematicians such as Hamilton, Kirchhoff or Cayley.
  - The core of graph theory focused on the study of regular graphs:
    
    Regular graph: a graph where all nodes have the same degree.
    
    Lattice: a regular network where all nodes are coupled to its nearest neighbor.

    \[ N = \text{number of nodes} \]
    \[ K = \text{degree} \]
    \[ C = \text{clustering coefficient} \]
    \[ d = \text{dimension of the lattice} \]
    \[ l = \text{average path length} \]

    \[
    C = \frac{3(K - 2d)}{4(K - d)} \\
    (\text{if } K < 2N/3)
    \]
    \[
    l \sim d \sqrt{\frac{N}{K}}
    \]
1.4.- BRIEF HISTORICAL BACKGROUND

- Paul Erdös (Budapest 1913 - Warsaw 1996)

  Some revealing data about Paul:

- Seminal contributions in combinatorics, graph theory, number theory, classical analysis, approximation theory, set theory, and probability theory.

- Paul wrote 1475 papers and collaborated with 511 scientists.

- Excentric person, he had an special vocabulary (children=“epsilons”, women=“bosses”, U.S=“samland”, etc...)

- Paul offered small prizes for solutions to unresolved problems (from 25$ to some thousands), and there are still open problems!

- “You don’t have to believe in God, but you should believe in The Book.” (he recognized that he took amphetamines)
1.4.- BRIEF HISTORICAL BACKGROUND

- **Paul Erdös and Alfred Rényi**

They worked on the analysis of social networks by finding analogies with the so-called *random graphs*, in which the existence of a link between a pair of nodes has a probability $p$.

- **Notations**
  - $N =$ number of nodes
  - $<k> =$ mean degree
  - $<L> =$ number of random connections
  - $p =$ probability of connection between two nodes

- **Formulas**
  - Mean degree of the network $\rightarrow <k> = p(N-1) \cong pN$
  - Number of random connections $\rightarrow <L> = \frac{1}{2} pN(N-1) \cong \frac{1}{2} <k>N$
1.4. - BRIEF HISTORICAL BACKGROUND

- Emergence of a giant component

When probability $p$ crosses a critical value $p_c$, there emerge a giant component that contains an extensive fraction of the nodes in the network.

- Critical probability ($N \to \infty$):
  \[ p_c \sim \frac{\ln N}{N} \]

- Critical mean degree:
  \[ \langle k \rangle_c \sim \ln N \]

- Clustering coefficient:
  \[ C = p \approx \frac{\langle k \rangle}{N} \ll 1 \]
  \[ N=1000 \quad <k>=2 \quad C \approx 0.002 \]

- Average shortest path:
  \[ \ell \sim \frac{\ln N}{\ln \langle k \rangle} \]
  \[ N=1000000 \quad <k>=5 \quad \ell \approx 8.6 \]
1.4. - BRIEF HISTORICAL BACKGROUND

- **Stanley Milgram (New York 1933 - New York 1984)**

  Stanley Milgram was an American social psychologist most notable for his controversial studies on the obedience to authority.

  Some Stanley’s famous experiments:

  - The Milgram experiment
  - The lost-letter experiment
  - The small-world experiment
1.4.- BRIEF HISTORICAL BACKGROUND

- The small-world experiment

A group of people from Omaha (Nebraska) and Wichita (Kansas) was asked to send a letter to an unknown person in Boston (Massachusetts).

Basic Rule of the experiment:

- People should forward the letter to a person that they consider closer to the target person

Results of one experiment (in fact, there where several!):

- 232 out of 296 letters never reached the target
- 64 letters reached the target (with paths from 2 to 10)
- The average path length was .... 5.2 (steps)
1.4.- BRIEF HISTORICAL BACKGROUND

- It’s a small world! (que pequeño es el mundo!)

This is a big world

This is a small world

or in other words:

\[ d_{ij} \ll N \]
1.4.- BRIEF HISTORICAL BACKGROUND

- Let’s go back to Erdős:
  You can measure the distance with Paul Erdős.
  (http://www.oakland.edu/entp/)
- Mean Erdős number: ~5
- Largest Erdős number: ~13
1.4.- BRIEF HISTORICAL BACKGROUND

It’s a small world everywhere!

The small-world property has been reported in a large number of real networks of different origin.

<table>
<thead>
<tr>
<th>NETWORK</th>
<th>SIZE</th>
<th>$\langle k \rangle$</th>
<th>$\ell$</th>
<th>$\ell_{\text{rnd}}$</th>
<th>$C$</th>
<th>$C_{\text{rnd}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Movie actors</td>
<td>225 226</td>
<td>61.0</td>
<td>3.65</td>
<td>2.99</td>
<td>0.79</td>
<td>0.00027</td>
</tr>
<tr>
<td>2. Power grid</td>
<td>4 941</td>
<td>2.67</td>
<td>18.7</td>
<td>12.4</td>
<td>0.08</td>
<td>0.00054</td>
</tr>
<tr>
<td>3. WWW site level (undir.)</td>
<td>153 127</td>
<td>35.2</td>
<td>3.10</td>
<td>3.35</td>
<td>0.11</td>
<td>0.00023</td>
</tr>
<tr>
<td>4. Words (co-occurrence)</td>
<td>460 902</td>
<td>70.1</td>
<td>2.67</td>
<td>3.03</td>
<td>0.44</td>
<td>0.00015</td>
</tr>
<tr>
<td>5. LANL co-authorship</td>
<td>52 909</td>
<td>9.70</td>
<td>5.90</td>
<td>4.79</td>
<td>0.43</td>
<td>0.00018</td>
</tr>
<tr>
<td>6. MEDLINE co-authorship</td>
<td>1 520 251</td>
<td>18.1</td>
<td>4.60</td>
<td>4.91</td>
<td>0.07</td>
<td>0.00001</td>
</tr>
<tr>
<td>7. Math. co-authorship</td>
<td>70 975</td>
<td>3.90</td>
<td>9.50</td>
<td>8.21</td>
<td>0.59</td>
<td>0.00005</td>
</tr>
</tbody>
</table>

Average path length and clustering coefficient of some real networks. We compare the values in the real network with those of equivalent random networks.

The average path length is similar in random networks (where $l \sim \ln N$) but the clustering coefficient is some orders of magnitude higher (and closer to the clustering coefficient of a lattice!).
1.4.- BRIEF HISTORICAL BACKGROUND

- The Watts-Strogatz model (I)

Watts and Strogatz (PRL 1998) proposed a network model that conciliated the high clustering and short average path length of real networks.

Starting from a regular ring, a certain (random) rewiring is introduced with a probability $p$.

$p$ increases
1.4.- BRIEF HISTORICAL BACKGROUND

The Watts-Strogatz model (II)

Small-world networks are characterized by a low average shortest path and high clustering.

A low number of “shortcuts” reduces the distance between nodes without modifying the local properties.
1.4.- BRIEF HISTORICAL BACKGROUND

- The Watts-Strogatz model (II)

The larger the network, the higher probability to be small-world.

The rewiring of the links in order to enter the small world-region goes with:

\[ p \sim \frac{1}{N} \]

Figure from Barthelemy, PRL, 82,3180 (1999)
1.4.- BRIEF HISTORICAL BACKGROUND

- The Watts-Strogatz model (III)

The probability degree distribution $p(k)$ of WS small-world networks shows a pronounced peak around $<k>$ and exponential decay.

Networks obtained with the WS model are “exponential networks”
1.4.- BRIEF HISTORICAL BACKGROUND

- **Scale-free networks (I)**

Unfortunatelly (or luckily!) many real networks are not exponential. On the contrary, they have a power-law decay (i.e., $P(k) \sim k^{-\gamma}$).

- Scale-free networks have power law decays $P(k) \sim k^{-\gamma}$

- Power laws are relatively slow decreasing functions (the probability of having highly connected nodes is much higher than in exponential networks).

- A power-law distribution has no peak at its average value (no characteristic scale).
1.4.- BRIEF HISTORICAL BACKGROUND

- Scale-free networks (II)

<table>
<thead>
<tr>
<th>NETWORK</th>
<th>SIZE</th>
<th>$\gamma_{in}/\gamma_{out}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Movie actors [57]</td>
<td>212 250</td>
<td>2.3</td>
</tr>
<tr>
<td>2. WWW [59]</td>
<td>$2 \cdot 10^8$</td>
<td>2.7/2.1</td>
</tr>
<tr>
<td>3. Internet, router [60]</td>
<td>260 000</td>
<td>—/1.94</td>
</tr>
<tr>
<td>5. Neuro. co-authorship [61]</td>
<td>209 293</td>
<td>2.1</td>
</tr>
<tr>
<td>6. SPIRES co-authorship [48]</td>
<td>56 627</td>
<td>1.2</td>
</tr>
<tr>
<td>7. E-mail messages [62]</td>
<td>59 912</td>
<td>1.5/2.0</td>
</tr>
<tr>
<td>8. Metabolic network [63]</td>
<td>778</td>
<td>2.2</td>
</tr>
</tbody>
</table>

Real networks with scale-free structure. From Almendral, PhD. Thesis

Interestingly, the exponent of the power laws range from 1.2 to 3, with the majority between 2 and 3.
1.4.- BRIEF HISTORICAL BACKGROUND

The Barabási-Albert model (I)

They introduce a model in order to explain the origin of the power-law distributions of real networks. A network is constructed from scratch following two fundamental rules:

- **Growth.** From an initial number of nodes $N_0$, new nodes are attached to the existing ones at discrete time steps. Thus, the number of nodes increases with time $N(t) = N_0 + t$ and also the number of links $L(t) = mt$ (being $m$ the number of links of each new node).

- **Preferential attachment.** The nodes to which the new node is attached are chosen following a preference function:

$$p_i = \frac{k_i}{\sum_{j=1}^{N(t)} k_j}$$
1.4.- BRIEF HISTORICAL BACKGROUND

- The Barabási-Albert model (II)

The BA model shows a power law decay independent of the number of links or the system size (with an exponent $\gamma=3$)

(Left) Degree distribution of the B-A model, with $N=m_0+t=300000$ and $m_0=1, 3, 5, 7$. The dashed lines correspond to $P(k)=k^{-2.9}$. (Right) $P(k)$ for $m_0=5$ and different systems size: $m=100000, 150000$ and $200000$. From R. Albert et al., Rev. Mod. Phys. 74, 47(2002).
1.4.- BRIEF HISTORICAL BACKGROUND

The Barábasi-Albert model (III)

As in random networks, the clustering coefficient obtained with the BA model is low.

Clustering coefficient $C$ of the network as a function of the system size $N$. From R. Albert et al., Rev. Mod. Phys. 74, 47 (2002)

<table>
<thead>
<tr>
<th>network</th>
<th>type</th>
<th>$n$</th>
<th>$m$</th>
<th>$C$</th>
<th>$\ell$</th>
</tr>
</thead>
<tbody>
<tr>
<td>film actors</td>
<td>undirected</td>
<td>449,913</td>
<td>25,516,482</td>
<td>0.78</td>
<td>3.48</td>
</tr>
<tr>
<td>company directors</td>
<td>undirected</td>
<td>7,673</td>
<td>55,392</td>
<td>0.88</td>
<td>4.60</td>
</tr>
<tr>
<td>math coursework</td>
<td>undirected</td>
<td>253,339</td>
<td>496,489</td>
<td>0.34</td>
<td>7.57</td>
</tr>
<tr>
<td>physics coursework</td>
<td>undirected</td>
<td>52,909</td>
<td>245,900</td>
<td>0.56</td>
<td>6.19</td>
</tr>
<tr>
<td>biology coursework</td>
<td>undirected</td>
<td>1,520,251</td>
<td>11,803,064</td>
<td>0.60</td>
<td>4.92</td>
</tr>
<tr>
<td>telephone call graph</td>
<td>undirected</td>
<td>47,000,000</td>
<td>80,000,000</td>
<td></td>
<td></td>
</tr>
<tr>
<td>email messages</td>
<td>directed</td>
<td>59,912</td>
<td>86,900</td>
<td>0.16</td>
<td>4.95</td>
</tr>
<tr>
<td>email address books</td>
<td>directed</td>
<td>16,881</td>
<td>57,029</td>
<td>0.13</td>
<td>5.22</td>
</tr>
<tr>
<td>student relationships</td>
<td>undirected</td>
<td>573</td>
<td>477</td>
<td>0.001</td>
<td>16.01</td>
</tr>
<tr>
<td>sexual contacts</td>
<td>undirected</td>
<td>2,810</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Internet</td>
<td>undirected</td>
<td>10,697</td>
<td>31,992</td>
<td>0.39</td>
<td>3.31</td>
</tr>
<tr>
<td>power grid</td>
<td>undirected</td>
<td>1,941</td>
<td>6,594</td>
<td>0.080</td>
<td>18.99</td>
</tr>
<tr>
<td>train routes</td>
<td>undirected</td>
<td>587</td>
<td>19,603</td>
<td>0.69</td>
<td>2.16</td>
</tr>
<tr>
<td>software packages</td>
<td>directed</td>
<td>1,439</td>
<td>1,723</td>
<td>0.082</td>
<td>2.42</td>
</tr>
<tr>
<td>software classes</td>
<td>directed</td>
<td>1,377</td>
<td>2,213</td>
<td>0.012</td>
<td>1.51</td>
</tr>
<tr>
<td>electronic circuits</td>
<td>undirected</td>
<td>24,097</td>
<td>53,248</td>
<td>0.030</td>
<td>11.05</td>
</tr>
<tr>
<td>peer-to-peer network</td>
<td>undirected</td>
<td>880</td>
<td>1,296</td>
<td>0.011</td>
<td>4.28</td>
</tr>
<tr>
<td>metabolic network</td>
<td>undirected</td>
<td>765</td>
<td>36,866</td>
<td>0.67</td>
<td>2.56</td>
</tr>
<tr>
<td>protein interactions</td>
<td>undirected</td>
<td>2,115</td>
<td>2,240</td>
<td>0.071</td>
<td>6.80</td>
</tr>
<tr>
<td>marine food web</td>
<td>directed</td>
<td>135</td>
<td>598</td>
<td>0.23</td>
<td>2.05</td>
</tr>
<tr>
<td>freshwater food web</td>
<td>directed</td>
<td>92</td>
<td>997</td>
<td>0.087</td>
<td>1.90</td>
</tr>
</tbody>
</table>

1.4. BRIEF HISTORICAL BACKGROUND

- The Barabási-Albert model (IV)

Attractiveness, aging, capacity, ... can modify the scale free behaviour of the BA model.

**The Dorogovtsev-Mendes-Samukhin model**

\[
\prod_{j \rightarrow i} = \frac{k_i + k_0}{\sum_l (k_l + k_0)}
\]

- \(k_0\) = initial attractiveness \((-m < k_0 < \infty)\)
- \(m\) = number of new links

\[
\gamma = 3 + \frac{k_0}{m}
\]

(\(2 < \gamma < \infty\))

Dorogovtsev et al.,
PRL 85 4633 (2000)

**The Kaprivsky et al. model**

\[
\prod_{j \rightarrow i} = \frac{k_i^\alpha}{\sum_l k_l^\alpha}
\]

- \(\alpha < 1\) : stretched exponential decay
- \(\alpha > 1\) : a single node dominates

Krapivsky et al.,
PRL, 4629 85 (2000)

**The Dorogovtsev-Mendes model**

Probability of linking depends on \(\tau^\alpha\)

(being \(\tau\) the age of the node)

Probability distribution for several aging exponents:
1) 0.2, 2) 0.25, 3) 0.5 and 4) 0.75. \(\alpha > 1\) exponential decay.
From PRE62, 1842 (2000)
1.4. - BRIEF HISTORICAL BACKGROUND

Complex Networks time line:

- 1736: Euler
- 1941: Flory-Stockmayer
- 1951: Solomon-Rappaport
- 1956: Birth of graph theory
- 1960: Polymers
- 1967: Word analysis (power-law)
- 1972: Social Networks
- 1998: More complex graph theory
- 1999: Power law Internet
- 1999: Scale-free model
- 2000+: Lots of physicists enter the game

Applications of Complex Networks
(57 de 58)
1.4.- BRIEF HISTORICAL BACKGROUND

All of that is nice, but does it have applications in real networks?
Thanks for your attention

mañana más, pero no mejor!