In phase and antiphase synchronization of coupled homoclinic chaotic oscillators

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We numerically investigate the dynamics of a closed chain of unidirectionally coupled oscillators in a regime of homoclinic chaos. The emerging synchronization regimes show analogies with the experimental behavior of a single chaotic laser subjected to a delayed feedback. © *2004 American Institute of Physics.* [DOI: 10.1063/1.1628431]

A peculiar chaotic dynamics consisting of a train of almost identical spikes separated by erratic interspike intervals (ISI) results from the homoclinic return to a saddle focus (HC=homoclinic chaos). HC has been studied with reference to a CO_2 laser with feedback,[1] but its occurrence is rather general, from chemical systems[2] to biological neurons.[3] Recently, HC has been shown to be susceptible of delayed self-synchronization (DSS), whereby a chaotic subset of spikes confined within a delay time T_d is reapplied as a feedback to its own generator, yielding a perpetual repetition of that chaotic subset.[4] Modeling DSS implies adding delayed feedback terms to the standard equations of HC (Ref. 5) and hence the problem becomes high-dimensional and the corresponding solution is rather lengthy.[6] Here we show that, coupling unidirectionally without delay N HC sites, the intrinsic delay τ between the spike of an active site and the triggered spike on the next site yields an overall delay $T_d = N\tau$, provided the last site is coupled to the first one. Thus a ring of unidirectionally coupled HC systems is dynamically equivalent to a system with delayed feedback.

Collective phenomena in chains of chaotic oscillators have recently attracted a wide interest, for the variety of possible scenarios that can be found, and the analogies with biological systems. Experiments have been performed on arrays of optical systems,[7] electronic circuits,[8] neurons,[9,10] and chemical oscillators,[11] reporting different synchronization patterns, such as antiphase synchronization, and clustering.[12] In a theoretical work, the dynamics of a unidirectionally coupled ring of chaotic systems has been explored in view of possible applications to neural systems.[13] However, the need for accurate setting of control parameters has up to now limited experiments to a small number of oscillators, hence most works on large chains are only numerical.[14] On the other hand, an analogy between spatially extended systems and delayed systems has already been drawn.[15]

For convenience we report a stretch of the experimental time signal of the homoclinic intensity [Fig. 1(a)] as well as its 2D phase space projection [Fig. 1(b)] obtained by an embedding technique and representing a superposition of a long sequence of spikes. The data of Fig. 1 correspond to the output intensity of a CO_2 laser with feedback. Precisely, the output intensity is detected, coded as an amplitude voltage plus a dc bias, and applied to drive an intracavity loss modulator.[16] For suitable ranges of the two control parameters (gain and bias of the feedback amplifier) a homoclinic chaotic behavior occurs, as represented in Fig. 1.[1]

We realize that on each cycle the intensity returns to its zero value baseline of Fig. 1(a) and point O in Fig. 1(b) which represents a saddle node (SN) fixed point, then emerging with a large spike followed by a decaying spiral toward a saddle focus (SF). The escape from SF is represented by a growing oscillation, which appears as a chaotic tangle in Fig. 1(b). Notice that the chaotic region around the SF is very contracted in phase space but stretched timewise, vice versa the spike occurring when leaving SN takes a short time but is spread over a wide space region.

The chaotic characteristic of the interspike interval (ISI) is due to the chaotic permanence time t_s around the saddle focus SF,[17] whereas the permanence time t_0 on the baseline is a fixed refractory time corresponding to the heteroclinic connection to SN; it stabilizes the orbit away from the saddle focus. This is confirmed by the values of the local Lyapunov exponents in the two regions.[18] The virtue of this heteroclinic orbit between SF and SN, that for simplicity we have called homoclinic chaos (HC),[16] is that introducing in the laser intensity channel [Fig. 1(a)] a threshold which cuts off the chaotic background, the relevant information is contained in the time occurrence of the geometrical identical spikes,

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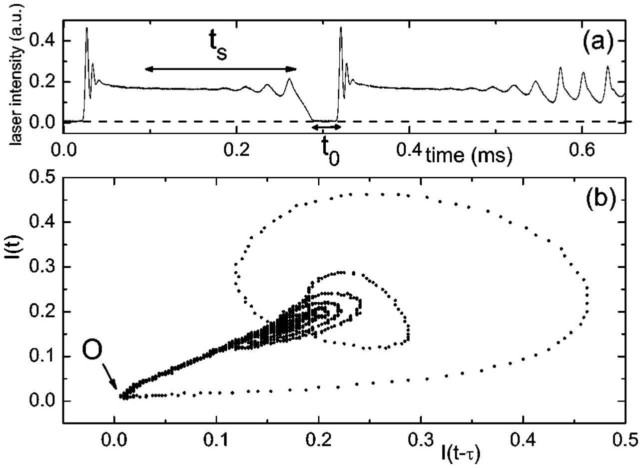

FIG. 1. (a) Experimental time intensity in the homoclinic regime. The dashed line corresponds to the baseline. (b) Phase space projection of the experimental time series.

which can be assimilated to Dirac delta functions of area S_0. In this representation, the useful signal is $S(t)=S_0\Sigma_i\delta(t-\tau_i)$, where τ_i is the time of occurrence of the ith spike, and $(\mathrm{ISI})_i=\tau_i-\tau_{i-1}$.

In these conditions, a delayed feedback can stabilize complex periodic orbits of period T. These orbits consist of a pseudochaotic train of pulses, that is, of a limited sequence with chaotic ISIs, that repeats after a time T again for ever. Of course T is chosen much larger than $\langle\mathrm{ISI}\rangle$, where $\langle\mathrm{ISI}\rangle$ is the ISI averaged over a long sequence.

We have called such a behavior as delayed self-synchronization (DSS) insofar as a chaotic sequence lasting for a time T_d, once reapplied as a feedback signal to the system, reproduces that sequence forever. This was demonstrated by the periodic occurrence of revival peaks in the autocorrelation function of the output intensity.[4]

This ability of DSS can be of interest in relation to recent studies on neuronal transformation mechanisms of short-term memories into permanent (long term) memories via the so-called synaptic reentry reinforcement.[19] Therefore, the possibility to describe DSS as a synchronized state in a closed chain of oscillators is a relevant issue.

First it is important to establish under what conditions a chain of coupled oscillators is equivalent to a delayed system. Since the delay implies that the information propagates in one direction, just unidirectional coupling will be considered in the oscillator chain, so that the oscillator at the site i is driven by the previous one at the site $i-1$. In addition, the delayed re-entry of the signal in the system means that the system is exposed to the total information generated over a previous time stretch of size T_d. Meeting this condition imposes a closed boundary with the last oscillator coupled to the first one.

With these boundary and coupling constraints, we build the array by using the scaled equations that model the experimental laser system,[5]

$$
\begin{aligned}
\dot{x}_1^i &= k_0 x_1^i (x_2^i - 1 - k_1 \sin^2 x_6^i),\\
\dot{x}_2^i &= -\gamma_1 x_2^i - 2k_0 x_1^i x_2^i + g x_3^i + x_4^i + p,\\
\dot{x}_3^i &= -\gamma_1 x_3^i + g x_2^i + x_5^i + p,\\
\dot{x}_4^i &= -\gamma_2 x_4^i + z x_2^i + g x_5^i + z p,\\
\dot{x}_5^i &= -\gamma_2 x_5^i + z x_3^i + g x_4^i + z p,\\
\dot{x}_6^i &= -\beta\left(x_6^i - b_0 + \frac{r(x_1^i - \epsilon x_1^{i-1})}{1+\alpha x_1^i}\right),
\end{aligned}
\tag{1}
$$

where the index i denotes the ith site position, and for each oscillator x_1 represents the laser intensity, x_2 the population inversion between the two resonant levels, x_6 the feedback voltage which controls the cavity losses, while x_3, x_4, and x_5 account for molecular exchanges between the two levels resonant with the radiation field and the other rotational levels of the same vibrational band.

In analogy with the experiment, the coupling on each oscillator has been realized by adding a function of the intensity (x_1) of the previous oscillator to the equation of its feedback signal x_6.

The parameters are the same for all elements of the chain. Here, k_0 is the unperturbed cavity loss parameter, k_1 determines the modulation strength, γ_1, γ_2, g are relaxation rates, p_0 is the pump parameter, z accounts for the number of rotational levels, β, b_0, r, α are, respectively, the bandwidth, the bias voltage, the amplification, and the saturation factors of the feedback loop, and ϵ is the coupling strength. The values used in the numerical simulation to reproduce the regime of homoclinic chaos,[5] are: $k_0=28.5714$, $k_1=4.5556$, $\gamma_1=10.0643$, $\gamma_2=1.0643$, $g=0.05$, $p_0=0.016$, $z=10$, $\beta=0.4286$, $\alpha=32.8767$, $r=160$, $b_0=0.1032$.

Besides the qualitative appearance of a chaotic tangle within the attractor, in close agreement with the experimental results of Fig. 1(b), the standard evaluation of Lyapunov spectrum provides evidence of one positive Lyapunov exponent.

The coupling strength ϵ is the control variable and it can assume both negative and positive values, as in the experiment. An important feature found in the experiment is that the route towards the pseudochaotic state depends on the sign of the delayed feedback modulation. If positive, the signal tends to reach in-phase synchronization with the delayed modulation, while if it is negative, the coupling comes out to be phase-repulsive and the signal is in antiphase with the feedback perturbation.

This route is not symmetric for the coupling strength: a significantly stronger coupling is needed to reach the pseudochaotic state for a positive than for a negative feedback signal. This behavior is a consequence of the fact that the system is more efficiently removed from the saddle focus neighborhood for a decrease of the modulating signal. Eventually, when the coupling is strong enough, the system becomes fully periodic for both positive and negative values, i.e., there is no longer a pseudochaotic status.

The experimental data[4] reveal a small time offset between the modulation and the signal, which is independent of the long delay T_d and of the coupling strength. This offset

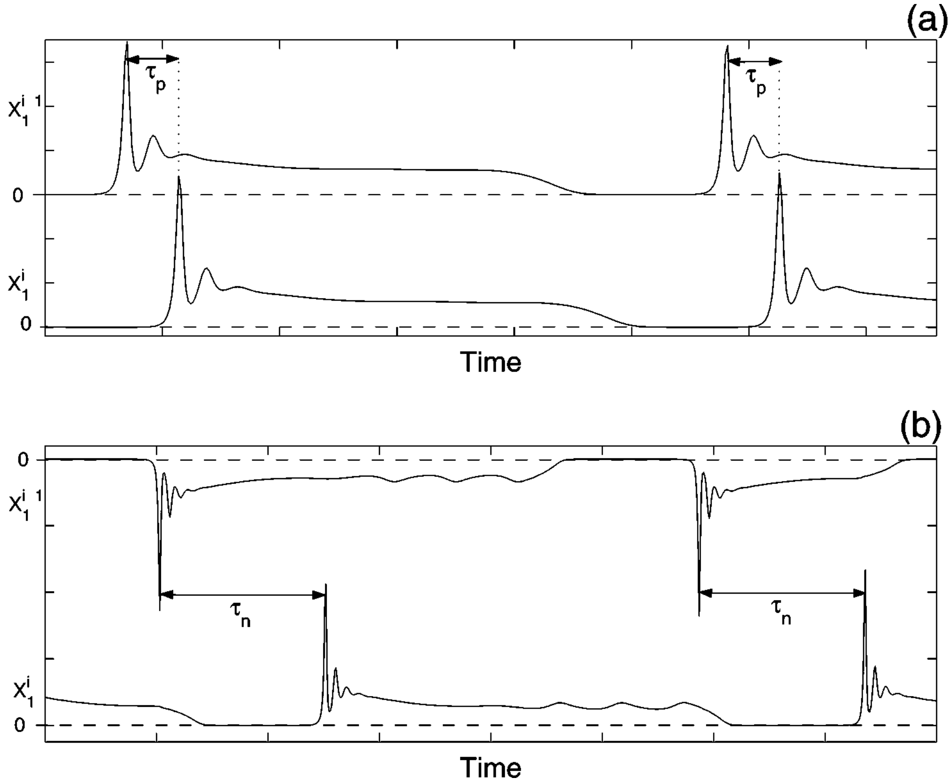

FIG. 2. Detail of the time profile of the ith (solid line) and $(i-1)$th elements of the chain, for (a) positive coupling and (b) negative coupling. The permanence time t_0 on the baseline is a fixed refractory time corresponding to the heteroclinic connection to SN.

depends on the sign of the modulation feedback, and it has been measured to be $\tau_n = 140\ \mu s$ for negative coupling and $\tau_p = 20\ \mu s$ for a positive, against an average ISI of 500 μs.

The numerical results are shown in Fig. 2, and explain how these asymmetries depend on the coupling sign. In both cases [positive (a) and negative (b) coupling], the slave intensity lags with respect to the driving one, but by fixed amounts, which however differ for different coupling signs. Precisely, the driver's negative slope forces the slave to escape away from the saddle region toward the zero baseline. In case (a), the driver's and slave's collapses onto the zero baseline occur just one after the other and the negative oscillations around the spike do not induce a transition insofar as they occur while the slave is already on the baseline. The offset τ_p depends on the coupling strength, that is, on how effective is the forcing to let the slave escape from SN. On the contrary, in case (b) the first negative driver's slope which can force the slave to fall away from the saddle coincides with the first negative spike oscillation; the escape of the slave from SF toward SN takes a lag time depending on the coupling strength; adding this lag to t_0 makes a total offset $\tau_n > \tau_p$.

Therefore we can say that the times τ_p and τ_n correspond to different information propagation velocities along the array, with a lower velocity (longer offset) for the negative coupling.

In order to model the delayed system with an array of N coupled systems, we must match the overall delay along the closed chain $N\tau_j$ (where j stands for p or n depending on the coupling) with a delay time T_d so that each system is exposed to a delayed version of its own signal $x_1(t-T_d) = x_1(t-N\tau_j)$. As $\tau_p < \tau_n$, a longer chain will be needed in the positive case to obtain the same periodicity. The scale of the different propagation velocities can be seen in Fig. 3, where the relation between the repetition time T and the number N of oscillators in the chain are reported for the two conditions. From Fig. 3 we can also establish that $\tau_n \simeq 7\tau_p$, which fits well with the experimental ratio 140/20. In order to adjust to the experiment, we had to choose different coupling strengths for the positive and the negative case, respectively, $\epsilon = 0.15$ and $\epsilon = -0.042$.

An example of the dynamical characteristics can be appreciated in Fig. 4. Here, the time intensity profile of a single site of the chain is compared with its driver neighbor in the pseudochaotic regime, for both positive [Fig. 4(a)] and negative [Fig. 4(b)] coupling. The chains have been chosen to have approximately the average period $\langle T\rangle = 0.95$ ms, and $\langle T\rangle = 1.25$ ms, respectively, with $N=7$ in the negative case and $N=70$ in the positive case.

We can thus establish a full equivalence between the experimental laser system of Ref. 4 and the closed chain here reported. As a pictorial demonstration, in Fig. 5 a chain of $N=8$ coupled negatively elements is compared to an experiment performed with negative feedback and a delay time of 4 ms, and another chain of $N=24$ elements with an experiment in which a delay of 12 ms was used. For the experimental case, we use the space–time representation of the laser intensity[15] [Figs. 5(a) and 5(b)], and report by horizontal black bars the occurrence of the largest spikes according to the previously described filtering process; for the numerical case we report for each site the occurrence of the largest spikes as a function of time [Figs. 5(c) and 5(d)]. The experimental results correspond to a negative coupling of -7%, while the numerical results are obtained for $\epsilon = -0.042$, these values being close to the threshold values to establish a stable pattern in the dynamics.

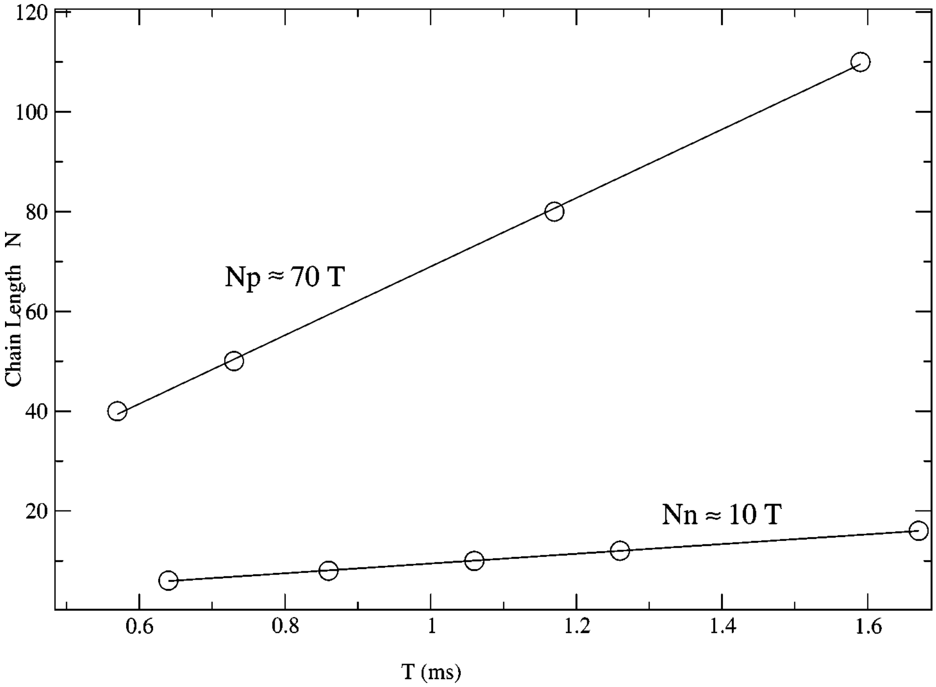

FIG. 3. Different responses of the chain for negative (Nn) and positive (Np) coupling. The overall propagation velocities N_j/T ($j=n,p$) are in the ratio 1/7 as the two offset times in Fig. 2.

In both theoretical and experimental cases we represent the occurrence of a spike by a short horizontal segment. The synchronization here is meant in its identical fashion,[12] and it has been checked by means of the vanishing of the synchronization error function.

The same phenomenology in which a positive coupling can induce a collective synchronous behavior, while a negative coupling induces an antiphase dynamics, has been observed in experiments in arrays of neurons where the coupling could be modified.[10]

In conclusion, we have discussed under what conditions a time delayed system can be used for an experimental investigation of the synchronization behavior of an array of chaotic oscillators. For a delayed laser in the homoclinic chaotic regime, phase and antiphase pseudochaotic synchronization regimes have been reported. The presented equivalence is crucial for experimental investigations, where a single system, but not a whole array, may be easily accessible; vice versa a computer simulation on a coupled array can be faster than the simulation of the delayed dynamics.

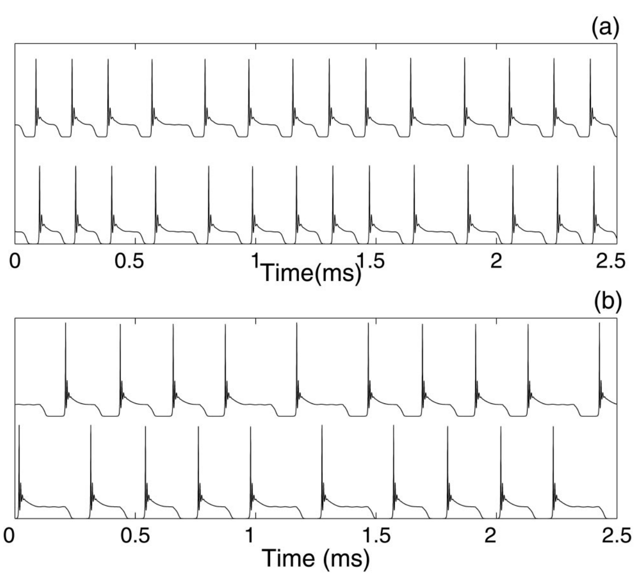

FIG. 4. Intensity profiles of two neighbor oscillators (a) $N=70$, $\epsilon=0.15$, (b) $N=7$, $\epsilon=-0.042$. The data have been vertically shifted for a better display.

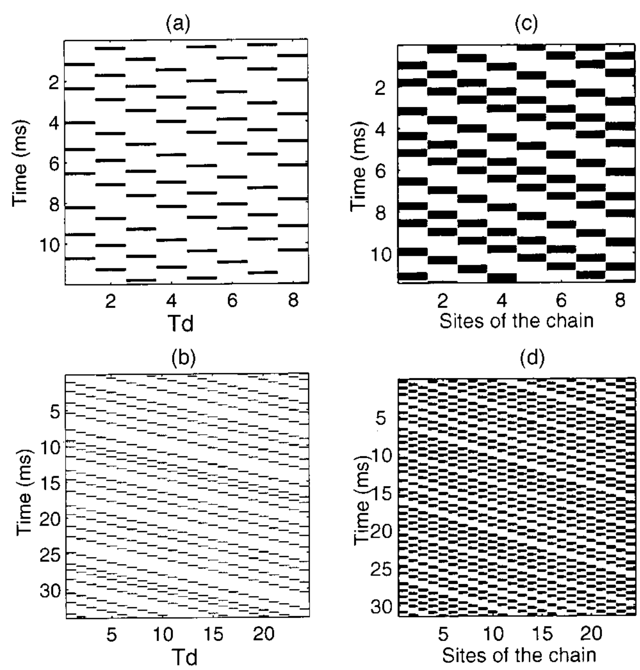

FIG. 5. Space–time representation of the experimental laser intensity in the experiments of Ref. 4 (a,b) compared with the numeric result of an oscillator chain (c,d). (a) $T_d=4$ ms, coupling strength -7%; (b) $T_d=12$ ms, coupling strength -7%. For the equivalent chain of oscillators: (c) $N=8$, $\epsilon=-0.042$; (d) $N=24$, $\epsilon=-0.042$.

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