

## Dynamical and spectral properties of complex networks

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**Abstract.** Dynamical properties of complex networks are related to the spectral properties of the Laplacian matrix that describes the pattern of connectivity of the network. In particular we compute the synchronization time for different types of networks and different dynamics. We show that the main dependence of the synchronization time is on the smallest nonzero eigenvalue of the Laplacian matrix, in contrast to other proposals in terms of the spectrum of the adjacency matrix. Then, this topological property becomes the most relevant for the dynamics.

### Contents

<b>1. Introduction</b>	<b>2</b>
<b>2. Linear dynamics</b>	<b>2</b>
<b>3. Spectral analysis</b>	<b>6</b>
<b>4. Nonlinear dynamics</b>	<b>7</b>
4.1. Kuramoto oscillators . . . . .	8
4.2. Majority dynamics. . . . .	9
<b>5. Conclusions</b>	<b>12</b>
<b>Acknowledgments</b>	<b>13</b>
<b>References</b>	<b>13</b>

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## 1. Introduction

In the last decade we have witnessed an enormous effort towards understanding the complex patterns of connectivity that have been found in many natural, social or technological systems [1]–[5]. Once the systems are characterized from a topological point of view, it is the turn of the dynamical properties and relating both dynamic and static characterizations has become one of the hot topics in network theory in recent years. There can be many different dynamics implemented in networks, ranging from simple discrete state systems, like cellular automata or random boolean networks, to networks of units whose individual behaviour is already complex, as happens in samples of coupled chaotic units. But, when dealing with the emergent behaviour characteristic of complexity, one of the main issues is to discern between the effects related to the topology and the effects related to the dynamical rules of the units.

In this paper we want precisely to understand which is the main topological characteristic of a network (undirected and unweighted) that influences the dynamical response. By looking at particular dynamical rules of the individual nodes and at particular rules of interaction between the units, the goal is to see what do they have in common in terms of the static properties of the network. In particular we want to analyse the route of complex networks to synchronization, understood as a stationary state in which all the units are in the same state. Synchronization of complex networks has been widely analysed in the past [6], mainly in the context of the master stability formalism (see [7]–[13]) that studies the stability of the synchronized state. Other studies have focused on the behaviour of small structures (motifs), as for instance [14, 15], and some other recent publications analyse the behaviour along the complete evolution of the system [16]–[21], highlighting the relation between spectral, topological and dynamical properties of networks.

Synchronization is a general concept and not related to a particular type of dynamics; for this reason, in order to stress the dynamical significance of the network parameters we consider three different types of dynamical evolution of the units and of the interaction rules: linear dynamics as a general approximation when the variables describing the units states are close to each other and hence close to the synchronized state, Kuramoto dynamics [22] that have been widely used in physical and biological problems, and a generic model of spin-like units that could model interactions between individuals in a social network.

The outline of the paper is as follows. In section 2 we analyse the simplest case of linear interactions between units. In section 3 we compare the roles of the eigenvalues of the adjacency matrix and the Laplacian matrix, going back to the dynamics in section 4 where the two different nonlinear dynamical rules are considered. We end in section 5 with the conclusions of our work.

## 2. Linear dynamics

Synchronization is a generic feature of coupled identical dynamical systems. In recent years the physics community has been focusing on the effect of connectivity patterns that go far beyond the usual regular lattices or all-to-all neighbouring schemes. Nowadays we have become used to considering units as nodes of a graph that are linked to other units in a complex way. Each unit evolves following its own dynamics and they are coupled according to some rules. Under generic conditions the coupling tends to favour the synchronization of the units. The first theoretical attempt to analyse the stability of the synchronized state of a complex network was

made by Barahona and Pecora [7]. Keeping the formalism to a minimum they proposed a system that obeys the following set of equations of motion

$$\frac{dx_i}{dt} = F(x_i) - \sigma \sum_{j=1}^n L_{ij} H(x_j), \quad (1)$$

where  $F$  corresponds to the unit evolution and  $H$  stands for the coupling;  $\sigma$  is the coupling strength and  $L_{ij}$  is the Laplacian matrix, related to the adjacency matrix,  $A_{ij}$  by the following relation:

$$L_{ij} = k_i \delta_{ij} - A_{ij}. \quad (2)$$

This Laplacian matrix is symmetric with zero row-sum and hence all the eigenvalues are real and non-negative. The eigenvalues are ordered such that

$$0 = \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_N. \quad (3)$$

The number of zero eigenvalues is equal to the number of connected components. If we are concerned with synchronization as a global effect, we have to notice that it is only possible in systems with a single connected component and hence there will be a single zero eigenvalue, implying  $\lambda_2 > 0$ . In general, the following inequality is also fulfilled [23]

$$\lambda_N \leq 2k_{\max}, \quad (4)$$

where  $k_{\max}$  is the largest degree in the graph.

Barahona and Pecora show that the synchronized state is stable if  $\frac{\lambda_N}{\lambda_2} < \frac{\alpha_B}{\alpha_A}$ , where  $\alpha_A$  and  $\alpha_B$  are the lower and upper bounds, respectively, of the effective coupling  $\sigma\lambda_i$  in which the maximum Liapunov exponent is negative. This inequality involves a part that depends on the topology of the network, the eigenvalue ratio  $Q = \frac{\lambda_N}{\lambda_2}$ , and a part that depends on the dynamical properties of the functions  $H$ ,  $F$ , and the values of the variables in the synchronized state. Thus it could be concluded [13] that the synchronizability of the system, understood as the stability of the synchronized state, is enhanced if the ratio  $Q$  is as small as possible. Since the value of the largest eigenvalue  $\lambda_N$  depends mainly on the maximum degree of the network, the main dependence will be, according to [13], on the smallest nonzero eigenvalue  $\lambda_2$ , usually called the *spectral gap*.

In this paper, we want to perform an additional step in the direction of characterizing synchronization in complex topologies, and for this reason we propose, as an additional parameter, the time the system needs to synchronize. Obviously, this characterization will depend on many factors: the type of dynamics of the single units, how strong the coupling is, and how far the initial set-up is from the synchronized state. In order to simplify this picture and to analyse how this time depends on the topological properties we will consider the simplest case of dynamics and coupling, and in the next sections we will study more complex dynamics. We will assume that each unit has a constant driving  $F(x_i) = \text{constant}$ , the same for all the units, and hence we can fix it to zero by transforming to a moving frame of reference. Furthermore, we will consider that the coupling is linear, which is a good approximation when the values of the variables describing the system are close to each other. Thus we deal with the system of

differential equations

$$\frac{d\theta_i}{dt} = -\sigma \sum_j L_{ij} \theta_j, \quad i = 1, \dots, N. \quad (5)$$

We should remark that, although we introduce this set of differential equations as a first approach to the problem of synchronization in complex networks, (5) itself is interesting in the context of distributed systems where it is known as *consensus dynamics* [24], having a long history in the field of computer science.

The Laplacian matrix is related to the topological properties of the network and hence it is the only relevant dependence; since the coupling strength  $\sigma$  just fixes the time scale. There is another obvious dependence on the initial conditions that will be discussed later.

The solution of this system reads in terms of the normal modes  $\varphi_i$  [16]

$$\sum_j B_{ij} \theta_j = \varphi_i(t) = \varphi_i(0) e^{-\lambda_i t}, \quad i = 1, \dots, N, \quad (6)$$

where  $B_{ij}$  is the matrix of the transformation from the old coordinates to the new ones. Thus we are left with linear combinations of phases in the original coordinates that is equal to a term that depends on the initial conditions multiplied by an exponential that decays very fast in time according to the eigenvalues of the Laplacian matrix. For very large  $t$  the exponentials decay to zero and the only solution is that all the units become synchronized. We can then assume that, at large times, the phase difference in the original coordinates decays exponentially with the smallest eigenvalue  $\lambda_2$ .

Thus we can write

$$\theta_i - \theta_j = C e^{-\lambda_2 t}, \quad (7)$$

where  $C$  is an unknown constant that depends on the specific details of the network and on the initial configuration.

Formally, the time the system needs to achieve complete synchronization is infinite. Usually in computer simulations one establishes a relaxed synchronization condition. We say that two oscillators are synchronized if the cosine of their phase difference is very close to 1,

$$\cos(\theta_i - \theta_j) \geq 1 - \varepsilon, \quad (8)$$

which means that  $\theta_i - \theta_j \sim \varepsilon^{1/2}$  and we can write

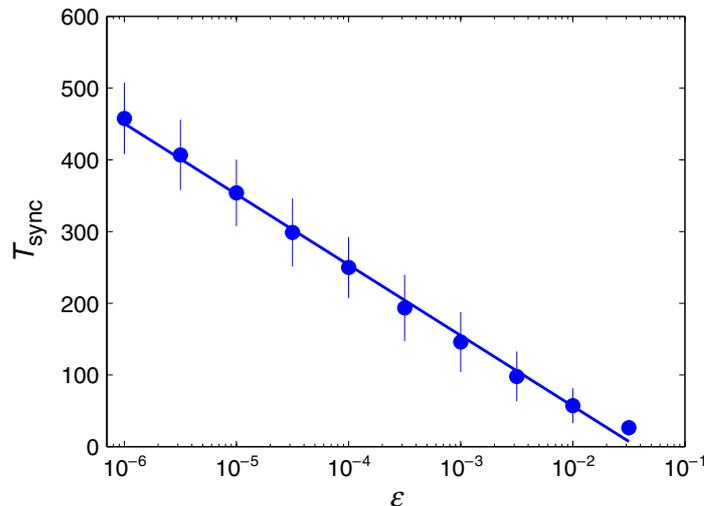
$$\frac{1}{2} \ln \varepsilon = \ln C - t \cdot \lambda_2 \quad (9)$$

and from here we can say that the synchronization time behaves in the following way:

$$T_{\text{sync}} \sim \frac{1}{\lambda_2} \left[ \ln C - \frac{1}{2} \ln \varepsilon \right]. \quad (10)$$

It is clear that this time depends on the topology and on the threshold condition.

In order to check these statements we have performed numerical simulations of (5) for different networks and thresholds. In all cases we have assumed random initial conditions



**Figure 1.** Time to synchronize as a function of  $\ln \epsilon$ . We have used a network of 256 nodes that was proposed in [16] as an example of network with two hierarchical community levels.

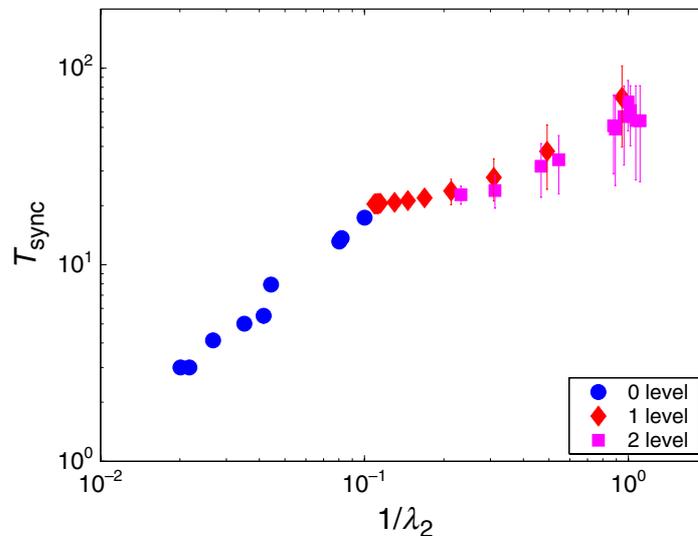
in the range  $[0, 2\pi]$ .<sup>5</sup> It is precisely this dependence on the initial conditions that makes the synchronization time  $T_{\text{sync}}$  a fluctuating magnitude. The range in the initial conditions is the responsible of the dispersion in the synchronization time, and this dispersion cannot be reduced by increasing the number of realizations.

We have considered different types of networks, with a wide range of topological features and sizes, just to focus on the dependence on the relevant characterization of the dynamical response of the network. Before entering into the details of the topology let us focus on the threshold dependence. To this purpose we consider a particular network and change the synchronization condition (8). The results of this set of simulations is plotted in figure 1, where we can observe a clear linear dependence of the synchronization time on  $\ln \epsilon$ , thus providing support to the assumptions we have made before. We have checked this dependence on other networks and dynamics and the conclusions are the same, the main dependence on the threshold is of the type shown in (10).

Following these assumptions now we want to check that, fixing the threshold condition, the synchronization time depends on the inverse of the spectral gap only. For this reason we have used networks with different sizes (128, 256 and 512), community structure at different hierarchical levels (1 level, 2 levels, and no communities), and growing rules. We have chosen networks grown according to different rules. Erdős–Rényi (ER) random graphs [25]; small-world models as proposed by Newman and Watts (NW) [26], in which the shortcuts are added instead of rewired as in the original Watts–Strogatz model [27]; and Barabási–Albert (BA) preferential attachment growing model networks [28].

Although these networks have different features, and this is reflected in their synchronization times, we want to stress that there is a clear dependence on the spectral gap, and this is indeed what can be concluded from figure 2, in which all the networks show an almost linear dependence on  $1/\lambda_2$ . Furthermore, although when moving from one class of network to another class of network

<sup>5</sup> This choice is due to the fact that in section 4, we will deal with phase oscillators where this is the natural choice.



**Figure 2.** Synchronization time for different networks. To show the dependence on the smallest nonzero eigenvalue of the Laplacian matrix, we have plotted the time to synchronize as a function of  $1/\lambda_2$ . In these plots, we fix the threshold to 0.99. We have used three sets of networks: networks introduced in [16] that have two community levels; networks introduced in [29] that have one community level; and networks with no community structure grown with different rules (see main text for details).

this dependence is not so clear, one can undoubtedly conclude that there is a monotonic increase of the synchronization time on the spectral gap. Networks with community structure need special care because neatly defined communities are related to very precise time scales for the internal synchronization within the community (see [16]) and, consequently, they show a different slope in figure 2.

This conclusion about the monotonic dependence on the spectral gap supports the previous analysis in [13] that highlights the role of this particular eigenvalue in the dynamical characterization of a complex network. In particular these authors show, apart from the stability of the synchronized state, that random walks propagate more easily in networks with large spectral gaps. This observation enables the authors to construct optimal graphs where the optimization goal is precisely the lowest spectral gap. In any case, we have shown that synchronization time depends mainly on this value for a set of linearly coupled dynamical systems. In section 4, we will come back to this issue dealing with other nonlinear dynamics but first we will discuss in the next section the role played by this eigenvalue and other proposals in the literature of complex networks.

### 3. Spectral analysis

In the previous section, we mentioned that, in recent literature, complex networks have been dynamically characterized by the spectral gap in terms of the stability of the synchronized state, following the original arguments of Barahona and Pecora [7], and also in terms of random

walks [13]. We have provided additional support by showing how the synchronization time strongly depends on this property. But also in recent literature some focus has been put on slightly different characterizations, in particular in [30] the authors propose the largest eigenvalue of the adjacency matrix. We have computed this eigenvalue  $\lambda_N^A$  and compared it with the spectral gap (the first nonzero eigenvalue of the Laplacian matrix). In general, for a complex network, there is no simple relation between the eigenvalues of the two matrices, the adjacency matrix and the Laplacian matrix. Only in the particular case of a regular lattice, or a network in which all nodes have exactly the same degree ( $k$ ), the eigenvalues satisfy the following relationship

$$\lambda_i^L = k - \lambda_{N-i+1}^A, \quad (11)$$

as can be easily concluded from 2 and keeping the same ordering for the eigenvalues of the adjacency matrix

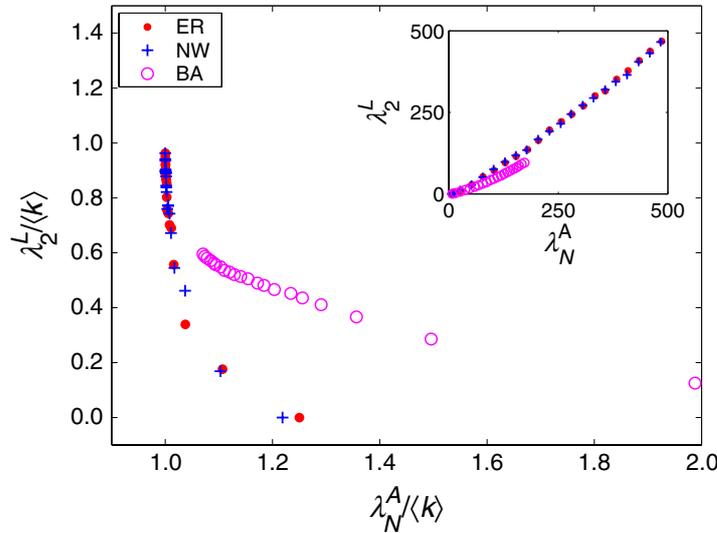
$$\lambda_1^A \leq \lambda_2^A \leq \dots \leq \lambda_N^A. \quad (12)$$

But if the distribution of degrees is not homogeneous, as usually happens in complex networks, then the relation is unknown. As a first approximation, for distributions of connectivities that are not far from the homogeneous one, as it happens for instance in random and in small-world graphs, we can still consider it. In this case, since the first eigenvalue of the Laplacian matrix is zero the largest eigenvalue of the adjacency matrix should be close to the average degree. Thus, networks that are quite homogeneous in degree will have a value of  $\lambda_N^A$  that is very close to the average degree, and hence it provides little information about the network structure and its dynamical properties. For this reason our proposal of characterizing the dynamical response of the system by the spectral gap is more appropriate.

In figure 3, we plot the second eigenvalue of the Laplacian matrix ( $\lambda_2^L$ ) against the largest eigenvalue of the adjacency matrix ( $\lambda_N^A$ ). There we can observe several facts that deserve some comments. Firstly, for networks with a homogeneous distribution of degrees, the largest eigenvalue of the adjacency matrix divided by the mean degree of the graph shows a very slight dependence on the network structure. On the contrary, the first nonzero eigenvalue of the Laplacian, divided by the mean degree as well, presents a more pronounced dependence on the network structure. Secondly, for networks with an inhomogeneous distribution of degree, such as the ones grown with the BA preferential attachment rule, both eigenvalues change with the network under consideration but they change in a similar fashion conserving a linear relationship. Then, one can conclude that, even in this case, there is no additional information in the adjacency matrix with respect to the Laplacian one. In summary, the first nonzero eigenvalue of the Laplacian matrix is more sensitive to network changes than the largest eigenvalue of the adjacency matrix, and for this reason it will be the focus of the next sections.

#### 4. Nonlinear dynamics

In order to check that the conclusions we have drawn before about the characterization of the dynamical properties of complex networks in terms of the spectral gap goes beyond the linear model, in this section we are going to consider two quite different nonlinear models: a model of phase oscillators and a model of spins. In both cases, according to the dynamics and the interaction, the system tends to synchronize. One of them synchronizes in a threshold sense, since phase is continuous, and the other in an absolute way since the spin states are discrete.



**Figure 3.** Smallest nonzero eigenvalue of the Laplacian matrix versus the largest eigenvalue of the adjacency matrix for different networks, both of them divided by the average degree of each network. Inset: the same but without normalization. We have chosen networks grown according to different rules. ER stands for Erdős–Rényi random graphs; NW for Newman–Watts [26], in which the shortcuts are added instead of rewired as in the original Watts–Strogatz model [27]; and BA for Barabási–Albert preferential attachment growing model [28].

#### 4.1. Kuramoto oscillators

One of the most studied models of coupled oscillators is that due to Kuramoto [22]. In this case oscillators follow the dynamics:

$$\frac{d\theta_i}{dt} = \omega_i + \sigma \sum_j A_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \dots, N, \quad (13)$$

where  $\omega_i$  stands for the natural frequency of the oscillator and  $\sigma$  describes the coupling between adjacent units. If the oscillators are identical ( $\omega_i = \omega \forall i$ ) there is again only one attractor of the dynamics: the fully synchronized regime where  $\theta_i = \theta, \forall i$ , which is stable. There has been a lot of effort in the recent literature on this model applied to complex networks [16]–[18], [31]. Concerning our current discussion relating spectral and dynamical properties one should notice [16] where the intermediate time scales are related to the topological scales of the networks which, in turn, are related to the distribution of eigenvalues. As we said before, the number of zero eigenvalues of the Laplacian matrix is equal to the number of connected components of the network. It is trivial then to conclude that if  $\lambda_2^L = 0$ , the network is split in more than one disconnected subnetworks. Then, from a dynamical point of view, it is impossible for the network to achieve a complete synchronized state, as only subnetworks with internal coherence but no synchronization between them are possible. Thus, a small value means that we are close to this situation and that it will take a long, although finite, time to synchronize completely (no matter how close to 1 the threshold condition is chosen).

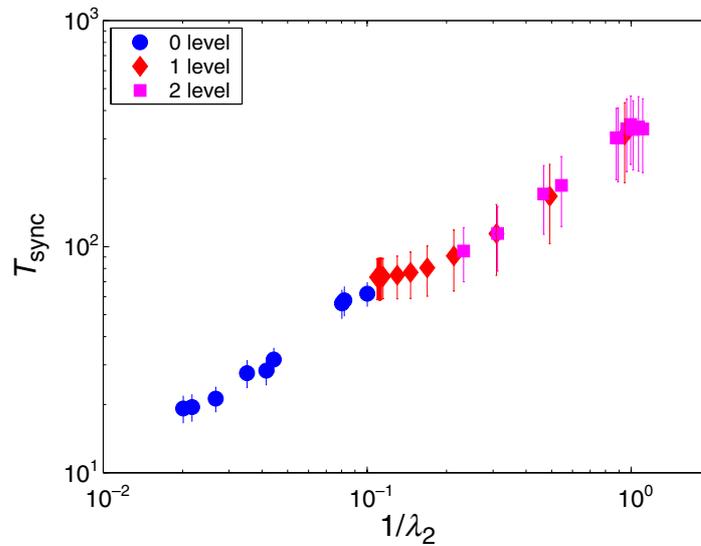
Actually, in [16, 17] it is shown that the existence of clearly defined communities [32], groups of nodes in which the number of internal links is large compared with the number of external ones, is related to gaps in the values of the eigenvalues, and the order of the gap is related to the number of communities, the sharper the community definition the larger the gap. This relation between spectral properties of the Laplacian matrix and the topological properties of the network is in turn reflected in the dynamics. Starting from a random distribution of phases and averaging over a set of distributions, what was observed is that synchronization appears from the innermost local scale to the outermost (albeit global) scale. In this synchronization process, the groups of nodes that get synchronized correspond to the topological communities and the times at which the groups merge to form larger groups are related to gaps in the spectrum of the Laplacian matrix. This relation closes the interdependence between topological, spectral and dynamical properties of the network. Then, in this general framework in which gaps in the spectrum are related to the achievement of synchronization at different scales, the last gap  $\lambda_2$ , always large compared with  $\lambda_1 = 0$ , should correspond to the completion of the synchronization process at the largest global scale.

With this goal in mind, we are going now to evaluate the synchronization time for a system of Kuramoto oscillators and analyse its dependence on the spectral gap. Before entering into the simulation details one has to notice that, starting the dynamics from a random distribution of phases, the oscillators rapidly settle into closer phases; after this fast initial evolution all phases are quite similar and the sine function in 13 can be well approximated by its argument. For this reason the linear model discussed in section 2 is a good approximation for the Kuramoto oscillators at later times, close to the synchronized state. This does not ensure, however, that this happens along all time evolution of the system and one has to be careful if the evolution concerns all time scales. In our numerical simulations, we have analysed the same type of networks as in section 2 for the Kuramoto dynamics and the results are shown in figure 4.

In figure 4, we can see that the time to synchronize for the different networks is around one order of magnitude larger; this is of course due to the initial rearrangement of the oscillators. Whereas in the linear model they are coupled linearly, in the Kuramoto model they are coupled through the sine function, which is smaller and makes the transient time needed to get similar phases larger. Once this transient is over, the phases are very similar and the two models behave in exactly the same way. Of course, if initial conditions in a small interval were chosen, the difference with respect to the linear model would be shortened, and the transient time reduced. In any case, the important point is that the scaling of the synchronization time with the spectral gap is identical in both models, enhancing our assumptions about the importance of the spectral gap as the key characterization of the dynamical response of the system. As happens with the linear interaction rules, here the dependence on the inverse of the spectral gap is not linear for the whole set of networks. Although the internal structure of the networks with communities can introduce important effects in the route towards global synchronization and break slightly the linear dependence, the monotonously increasing behaviour with  $1/\lambda_2$  is maintained.

#### 4.2. Majority dynamics

The linear model analysed in section 2 and the Kuramoto model discussed in the previous paragraph are described by continuous variables, phases, and the synchronization is understood in a threshold sense. In any case we have shown that the synchronization process is similar in both cases, being quite different in the transient to synchronization but very similar when arriving



**Figure 4.** Synchronization time for the same networks as in figure 2, for dynamical evolution given by 13. All the details of the simulations, such as distribution of random phases, threshold for synchronization, and coupling constant, are the same.

at the synchronized state. In this subsection we propose a completely different model, in which the dynamics is discrete, and hence does not share the processes described above.

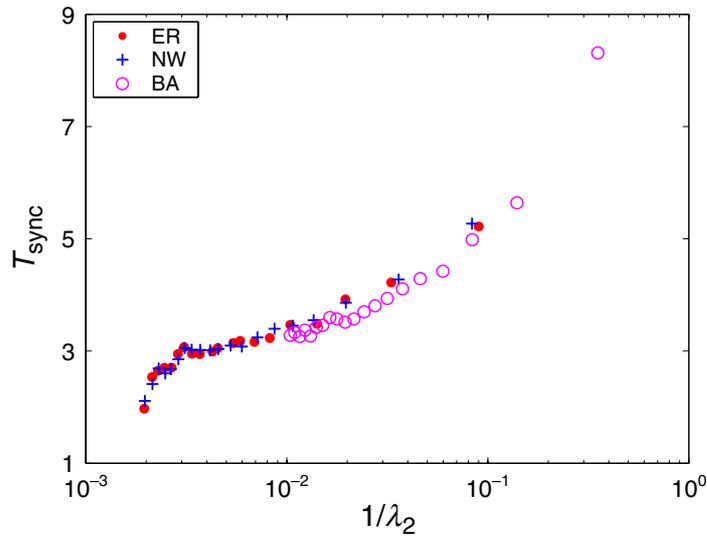
Let us consider a discrete spin-like system in which the nodes of a network have only two possible states,  $s_i = \pm 1$ . This could model, for instance, the dynamics of public opinion in social influence networks (e.g. when a group of people choose among two different opinions). Initially, half of the spins are randomly set at the state  $-1$  and the other half at  $+1$ . Then, each node  $i$  receives an input  $h_i = \sum_j A_{ij}s_j$ , with  $A_{ij}$  being the adjacency matrix. In this manner, as other authors have pointed out [33, 34], this spin-like network is characterized at each step by some pattern of internal states, whose evolution represents the global dynamics.

We evolve the network according to the following local majority rule: the state of node  $i$  at time  $t + 1$  is given by

$$s_i(t + 1) = \begin{cases} +1 & \text{if } h_i(n) > 0, \\ s_i(t) & \text{if } h_i(n) = 0, \\ -1 & \text{if } h_i(n) < 0. \end{cases} \quad (14)$$

We find that the system does not synchronize for some initial states. That is, the system wanders in the phase space without reaching a fixed point. This phenomenon is a kind of frustration in which the system is unable to reach the lowest energy state.

To focus our attention on how the topology contributes to synchronizing the system, we overcome this frustration by introducing a slight perturbation, which can be regarded as noise or a thermal bath. We find that it is enough, for the system to synchronize, that 0.5% of the states are randomly switched at every step. The introduction of such a perturbation has a drawback, in the sense that it also destroys the final synchronized state. For this reason we will consider that the system is synchronized if 99% of the nodes are in the same discrete state.



**Figure 5.** Synchronization time for a discrete spin-like dynamics following a local majority rule. Three types of networks are considered: ER, NW, and BA. In all of them we find that the synchronization time monotonically grows with  $1/\lambda_2$ .

According to these rules we have performed several numerical simulations to compute the average time required to synchronize networks, all of them formed by 512 nodes, grown again following different models. Namely, we have considered several ER, NW and BA networks.

Since it is impossible to synchronize a system with disconnected components, we have only studied networks with a unique connected component and refused those that do not verify such condition. This is not a problem for NW and BA networks because their growth is such that all nodes are linked. But this is not the case for ER networks in general and we have only considered ER networks with a single component.

We observe that the synchronization time monotonously depends on the inverse of the spectral gap as we have found in the previous simulations (figure 5). The importance of this particular eigenvalue is again clear, although in the present case the functional dependence is different from the previous continuous models (linear and Kuramoto). Nevertheless, due to the fact that the dynamical rules are completely different, it gives more arguments to our line of reasoning and the spectral gap should be considered the main characteristic of the network concerning the dynamical response.

In principle, the dynamical evolution of this spin-like system is quite different from the linear and Kuramoto models analysed above. However, the dynamical evolution can be written in compact form

$$s_i(t+1) = \Theta \left[ \sum_j A_{ij} s_j(t) + \mu s_i(t) \right], \quad (15)$$

where  $\Theta(x)$  is the Heaviside step function, and  $\mu$  is a number  $0 < \mu < 1$  that ensures that there is no change in the case of a tie between the neighbours. In this equation one can notice that the

argument of the Heaviside function can be written in matrix form

$$\sum_j A_{ij} s_j(t) + \mu s_i(t) = \sum_j \left[ A_{ij} + \frac{\mu}{k_i} \delta_{ij} \right] s_j(t). \quad (16)$$

In this way the evolution of the units reads

$$s_i(t+1) = \Theta \left[ \sum_j \left[ A_{ij} + \frac{\mu}{k_i} \delta_{ij} \right] s_j(t) \right], \quad (17)$$

in which two important facts should be noticed. Firstly, the Heaviside function imposes a quite strong nonlinearity that, eventually, could be regularized but, in principle, can be responsible for the rapid convergence towards the synchronized state of this dynamical rule. Secondly, the matrix in the argument

$$A_{ij} + \frac{\mu}{k_i} \delta_{ij}, \quad (18)$$

plays a key role in the way the coupling is performed. This matrix, which can be related to the Laplacian and adjacency matrices, can be analysed in the same terms as these others. Since its eigenvalues should give information about the dynamical processes taking place, its calculation and comparison with the other spectral properties becomes relevant and it will be the objective of future work.

## 5. Conclusions

In this paper we have presented results on different types of dynamics running in complex networks. We propose a new dynamical measure to characterize the dynamical properties of networks, the synchronization time. Although this time can depend on many factors, mainly the type of dynamics that is implemented and on others like the coupling and the initial conditions, we observe that this time basically depends on one of the static features of the network, the so-called spectral gap, the smallest nonzero eigenvalue of the Laplacian matrix. The role played by this eigenvalue has also been stressed in the features of complex networks related to the dynamics. We have also compared this eigenvalue with another proposal from the literature, based on the spectral properties of the adjacency matrix, and we have found that the dependence is clearer in terms of the inverse of the spectral gap of the Laplacian matrix. Likewise, we have found that different dynamics can be described in terms of other matrices, different from the traditional studies based on the Laplacian or adjacency matrices.

The study we have performed relating the spectral, topological and dynamical properties of complex networks has an immediate continuation in terms of the robustness of the network. Usually robustness is defined in terms of the topology, i.e. how the network connectivity responds to external attacks, but we are convinced that relating topological and dynamical properties would give more hints about the dynamical robustness of the network, which is the dynamical response of the network to dynamical attacks.

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